CONTENT

- Length contraction
- Time dilation
- Addition of velocities

http://nptel.ac.in/courses/115101011/6

Length contraction

In classical mechanics, the length of an object is independent of the velocity of the moving observer relative to the object. However, on the basis of the theory of relativity, the length of an object depends upon the velocity of the observer with respect to the object.

Let us consider two inertial systems F and F′. The system F′ is moving with velocity v relative to the system F along X-axis, Let a rod AB be at rest in moving system F′ relative to the observer O′ and L_0 be the length of the rod in this frame measured by observer O′ at any instant. This length L_0 measured from the system in which the rod is at rest is called proper length. So L_0 will be given by

$$L_0 = x_2' - x_1' \qquad \text{(i)}$$

where x_1' and x_2' are the coordinates of two ends of the rod at any instant. At the same time, the length of this rod (say L) measured by an observer O in the stationary frame F is given by

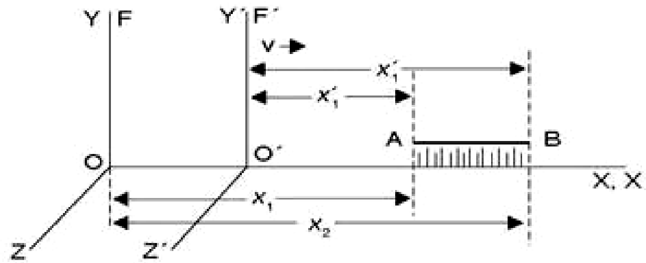
$$L = x_2 - x_1 \qquad \text{(ii)}$$

where, x_1 and x_2 are the abscissae of the ends of the rod in the frame F.

As per Lorentz transformation,

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \qquad \text{(iii)}$$

$$x_2' = \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} \qquad \text{(iv)}$$



Substracting Eq. (iii) from Eq. (iv) we have,

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

or $$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}} \qquad \text{(v)}$$

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \qquad \text{(vi)}$$

From Eq. (vi), we see that $L < L_0$. Thus, the length of the rod is reduced in the ratio $\sqrt{1 - v^2/c^2} : 1$ as measured by the observer moving with velocity v with respect to the rod.

Time dilation

The word dilation means to lengthen. Consider two coordinate systems F and F′ such that F′ is moving with velocity v along the X-axis relative to F. Imagnine a gun placed at the fixed position (x', y', z') in the frame F′. Suppose it fires two shots at time intervals t_1' and t_2' measured by observer O′ in the frame F′.

The time interval $(t_2' - t_1')$ of two shots measured by the clock at rest in moving frame F′ is called *proper time interval* and is given by

$$t_2' - t_1' = t_0 \qquad \text{(i)}$$

As the motion is relative, we may assume that F is moving with velocity $-v$ along the +X-axis relative to F′. In the frame F, the observer O, which is at rest, observes these two shots at different times t_1 and t_2. The time interval appears to him is given by

$$t_2 - t_1 = t \qquad \text{(ii)}$$

From inverse Lorentz transformation equations, we have

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{(iii)}$$

$$t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{(iv)}$$

By using Eqs. (iii) and (iv), we get,

$$t = t_2 - t_1$$

or $$t = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} \qquad \text{(v)}$$

Eq. (v) shows that $t > t_0$, i.e., the time interval appears to be lengthened by a factor $\frac{1}{\sqrt{1 - v^2/c^2}}$ which is observed by the observer O in frame F. This is known as *time dilation*.

Addition of velocities

The classical laws of addition of velocities need to be modified at very high velocities. Consider two frames of references F and F$'$ such that the frame F$'$ is moving with a velocity v relative to F along X-axis. Suppose a particle is moving relative to both the systems F and F$'$. Let u and u' be the velocities of the particle measured in frames F and F$'$, respectively. Then the velocity components are given as

$$\left.\begin{aligned} u_x &= \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}, \\ u_x{}' &= \frac{dx'}{dt'}, \quad u_y{}' = \frac{dy'}{dt'}, \quad u_z{}' = \frac{dz'}{dt'}, \end{aligned}\right\} \qquad (i)$$

From inverse Lorentz transformations, we have

$$x = \frac{x' + vt'}{\sqrt{1 - v^2 / c^2}} \cdot y = y', z = z' \quad \text{and} \quad t = \frac{t' + vx' / c^2}{\sqrt{1 - v^2 / c^2}} \tag{ii}$$

By differentiating these equations, we get

$$dx = \frac{dx' + vdt'}{\sqrt{1 - v^2 / c^2}} \cdot dy = dy', dz = dz', \; dt = \frac{dt' + \frac{vdx'}{c^2}}{\sqrt{1 - v^2 / c^2}} \tag{iii}$$

From Eqs. (i) and (iii), we have

$$u_x = \frac{dx}{dt} = \frac{dx' + vdt'}{dt' + \frac{vdx'}{c^2}} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'}$$

$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}} \tag{iv}$$

Similarly,

$$u_x = \frac{dy}{dt} = \frac{dy'\sqrt{1 - v^2 / c^2}}{dt' + \frac{vdx'}{c^2}} = \frac{\frac{dy'}{dt'}\sqrt{1 - v^2 / c^2}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \qquad [\because \quad dy = dy']$$

$$u_y = \frac{u_x'\sqrt{1 - v^2 / c^2}}{1 + \frac{vu_x'}{c^2}} \tag{v}$$

Similarly,

$$u_z = \frac{u_z'\sqrt{1 - v^2 / c^2}}{1 + \frac{vu_x'}{c^2}} \tag{vi}$$

Eqs. (iv), (v) and (vi) represent the relativistic laws of addition of velocities whereas in classical mechanics u_x is simply represented by

$$u_x = u_x' + v$$

If $u_x' = c$, i.e., if the light is emitted in the moving frame F′ along its direction of motion relative to F, then

$$u_x = \frac{u_x' + v}{1 + \frac{v u_x'}{c^2}} = \frac{c + v}{1 + \frac{vc}{c^2}} = \frac{c(c+v)}{(c+v)} = c$$

Thus, from the above expression it is clear that the speed of light is the same in all inertial frames.

If u_x' and v are smaller as compared to c, then $\frac{v u_x'}{c^2}$ can be neglected as compared to unity and u_x becomes $u_x = u_x' + v$, the law of addition of velocity which is similar to the one in classical mechanics.